ELEMENTS OF NUMBER THEORY & CONGRUENCES

Lagrange, Legendre and Gauss

Mathematics

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ELEMENTS OF NUMBER THEORY & CONGRUENCES

1) If \( a \neq 0, \ b \neq 0 \in \mathbb{Z} \) and \( a/b, \ b/a \) then

1) \( a=b \)
2) \( a=1 \)
3) \( b=1 \)
4) \( a=\pm b \)
Ans: is 4 known result.

If \(\frac{a}{b} \Rightarrow b = ma \Rightarrow (1)\) where \(m \in \mathbb{Z}\) & \(b/a \Rightarrow a = bn \Rightarrow (2)\) where \(n \in \mathbb{Z}\) from (1) & (2), \(a = (am)n = a(mn) \Rightarrow mn = 1\), possible if \(m=1 \& n=1\) or \(m=-1 \& n=-1\). For the values of \(n=1 \& -1\) then (2) \(\Rightarrow a = \pm b\)
2) 0 and 1 are

1) primes
2) composite numbers
3) neither prime nor composite
4) none of these
Ans: is 3 by defn. of prime & composite numbers it's implied
3) If \((ab,c) = 1 \& (a, c)=1\) then \((b, c)=\)

1) 1
2) c
3) b
4) none of the these
Ans: is 1

known result

\[(a, c) = 1, \ (b, c) = 1 \implies (ab, c) = 1\]
4) If p is prime number then p/ab ⇒

1) p/a
2) p/b
3) p/a or p/b
4) none of the these
Ans: is 3. known result

$p/ab \Rightarrow p/a \text{ or } p/b$
5) 111………1 (91 times) is

1) a composite number
2) a prime number
3) a surd
4) Irrational
Ans: is 1
since $91 = 7 \times 13$

$\underbrace{1111\ldots1}_{91 \text{ times}} = \underbrace{1111111}_{7 \text{ times}} \cdot \underbrace{1111111}_{7 \text{ times}}$ (13 factors)

$\therefore$ it is divisible by 1111111. (7 times)

$\therefore$ It is a composite number.
6) The number of positive divisors of 1400, including 1 and itself is

1) 18
2) 24
3) 22
4) 21
Ans: is 2

1400 = 2^3 \times 5^2 \times 7

\therefore T(1400) = (3+1) (2+1)(+1)

= 24
7) The sum of all positive divisors of 960 excluding 1 and itself is

1) 3047
2) 2180
3) 2087
4) 3087
Ans: is 3

960 = 2^6 \times 3 \times 5

\[ S(960) = \left( \frac{2^{6+1} - 1}{3 - 1} \right) \left( \frac{3^{1+1} - 1}{3 - 1} \right) \left( \frac{5^{1+1} - 1}{5 - 1} \right) \]

= 127 \times 4 \times 6 = 3048

but 3048 – 960 – 1 = 2087.
8) If \((a+b)^3 \equiv x \pmod{a}\) then

1) \(x = a^2\)
2) \(x = b^3\)
3) \(x = a^3\)
4) \(x = b^2\)
Ans: is 2

\[(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]

\[\Rightarrow (a+b)^3 - b^3 = a(a^2 + 3ab + 3b^2) = ak\]

\[\Rightarrow a / [(a+b)^3 - b^3]\]

\[\therefore (a+b)^3 \equiv b^3 \pmod{a}\]
9) Which of the following statement is false?

1) $98 \equiv -7 \pmod{3}$
2) $67 \equiv 2 \pmod{5}$
3) $123 \equiv -4 \pmod{7}$
4) $240 \equiv 9 \pmod{11}$
Ans : is 3

123 + 4 = 127 is not a multiple of 7
10) If $100 \equiv x \pmod{7}$, then the least positive value of $x$ is

1) 1
2) 3
3) 4
4) 2
Ans: is 4

\[
\frac{7}{(100 - x)} \text{ when } x = 2, \\
\frac{7}{98}
\]
11) When $5^{20}$ is divided by 7 the remainder is

1) 1
2) 3
3) 4
4) 6
Ans : is 3

\[ 5^3 = 125 \equiv -1 \pmod{7} \]

\[ \therefore (5^3)^6 \equiv (-1)^6 \pmod{7} \]

\[ 5^{18} \cdot 5^2 = 1 \cdot 5^2 \pmod{7} \]

\[ \therefore 5^{20} \equiv 25 \pmod{7} \equiv 4 \pmod{7} \]
12) The last digit in $7^{291}$ is

1) 1
2) 3
3) 7
4) 9
Ans : is 2

$$7^2 = 49 \equiv -1 \pmod{10}$$

$$\Rightarrow (7^2)^{145} \equiv (-1)^{145} \pmod{10}$$

$$7^{290} \equiv -1 \pmod{10}$$

also $$7 \equiv -3 \pmod{10}$$

$$\therefore 7^{190} \times 7 \equiv (-1)(-3) \pmod{10}$$

$$\therefore 7^{291} \equiv 3 \pmod{10}$$
13) The digit in the unit place of the number $183! + 3^{183}$ is

1) 7
2) 6
3) 3
4) 0
Ans : is 1

Unit place in $183!$ is 0 (∵ it is a factor of 10)

& $3^2 = 9 \equiv -1 \pmod{10}$

$(3^2)^{91} \equiv (-1)^{91} \pmod{10} = -1 \pmod{10}$

∴ $3^{182} \equiv -1 \pmod{10}$ also, $3 \equiv -7 \pmod{10}$

∴ $3^{182} \cdot 3 \equiv (-1) (-7) \pmod{10}$

∴ $3^{183} \equiv 7 \pmod{10}$
14) If \(-17 \equiv 3 \pmod{x}\), then \(x\) can take the value

1) 7
2) 3
3) 5
4) None of these

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Ans: is 3

-17 - 3 = -20 is divisible by 5
15) The smallest positive divisor of a composite integer a (>1) does not exceed

1) $a^2$
2) $\sqrt[3]{a}$
3) $a^3$
4) $\sqrt{a}$
Ans: is 4

Known result

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16) Which following linear congruences has no solution

1) $4x \equiv 1 \pmod{3}$
2) $3x \equiv 2 \pmod{6}$
3) $5x \equiv 3 \pmod{4}$
4) $2x \equiv 1 \pmod{3}$
Ans: is 2
Since \((3, 6) = 3\) & 3 does not divide 2
\[\therefore\] No solution
17) The relation congruence modulo \( m \) is

1) Reflexive
2) Symmetric
3) Transitive only
4) All of these

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Ans: is 4

Known result

\[ a \equiv b \pmod{m} \] is an equivalence relation

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18) The least positive integer to which 79 x 101 x 125 is divided by 11 is

1) 5
2) 6
3) 4
4) 8
Ans : is 1
79 ≡ 2 (mod 11), 101 ≡ 2 (mod 11)
& 125 ≡ 4 (mod 11) multiplying these,
79x101x125 ≡ 2x2x4 ≡ 16 (mod 11)
but 16 ≡ 5 (mod 11)
∴ 79 x 101 x 125 ≡ 5 (mod 11)
19) If $p \equiv q \pmod{m}$ if and only if

1) $(p - q) / m$
2) $m/(p - q)$
3) $m/p$
4) $m/q$
Ans: is 2
by very defn. Of congruence
i.e. if $a \equiv b \pmod{m} \Rightarrow m/(a-b)$
20) When $2^{100}$ is divided by 11, the remainder is

1) 3
2) 5
3) 1
4) 2

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Ans: is 3

$2^5 = 32 \equiv -1 \pmod{11}$

$\therefore (2^5)^{20} \equiv (-1)^{20} \pmod{11}$

$\therefore 2^{100} \equiv 1 \pmod{11}$
21) If \( a \equiv b \pmod{m} \) and \((a, m) = 1\), then

1) \((a, b) = 1\)
2) \((b, m) = 1\)
3) \((b, m) = a\)
4) \((a, b) = m\)
Ans: is 2

Known result

(a, m) = (b, m) = 1
22) If \( n \equiv 0 \pmod{4} \) then \( n^3 - n \) is divisible by

1) 6 but not 24
2) 12 but not 24
3) 24
4) 12 & 24
Ans : is 2

n is a multiple of 4

if n=4, \(n^3 - n = 60\)

\[\therefore 12/60, 6/60 \text{ but 24 does not divided by } 60\]

Thus 6 & 12 divide \(n^3 - n\).
23) If $195 \equiv 35 \pmod{M + 2}$ then
$m =$

1) 4
2) 5
3) 0
4) 7
Ans: is 3

\[(m+2) / (195-35) \Rightarrow (m+2) / 160\]

\[\Rightarrow m+2 \geq 2\]

\[\Rightarrow m+2 = 2, 4, 5, 8 \ldots \ldots \text{etc.}\]

\[\Rightarrow m = 0, 2, 3, 6 \text{ etc.,}\]

\[\therefore (3) \text{ is the answer}\]
24) If $2^8 \equiv (a+1) \pmod{7}$ is true then $a$ is

1) 3
2) 4
3) 0
4) 5

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Ans: is 1

\[ 2^6 = 64 \equiv 1 \pmod{7} \]

\[ 2^6 \cdot 2^2 = 1 \cdot 2^2 \pmod{7} \]

\[ 2^8 \equiv 4 \pmod{7} \]

\[ \Rightarrow a + 1 = 4 \text{ i.e., } (a = 3) \]
25) The unit digit in $13^{37}$ is

1) 5
2) 2
3) 6
4) 3
Ans: is 4

\[ 13^2 = 169 \equiv -1 \pmod{10} \]

\[(13^2)^6 \equiv (-1)^6 \pmod{10}\]

\[13^{36} \cdot 13 \equiv 1 \cdot 13 \pmod{10}\]

\[\therefore 13^{37} \equiv 3 \pmod{10}\]
26) The number of incongruent solutions of $24x \equiv 8 \pmod{32}$ is

1) 2
2) 4
3) 6
4) 8
Ans: is 4 by thm.

\[(24, 32) = 8 \& \frac{8}{8}\]

\[\therefore\text{ the number of incongruent solutions} = 8\]
27) The remainder when $3^{100} \times 2^{50}$ is divided by 5 is

1) 3
2) 4
3) 1
4) 2
Ans : is 2

\[ 3^2 = 9 \equiv -1 \pmod{5} \implies (3^2)^{60} = (-1)^{50} \pmod{5} \]

\[ \therefore 3^{100} \equiv 1 \pmod{5} \implies (1) \]

& \[ 2^2 = 4 \equiv -1 \pmod{5} \implies (2^2)^{25} = (-1)^{25} \pmod{5} \]

\[ \therefore 2^{50} \equiv -1 \pmod{5} \implies (2) \]

(1) x (2) \[ \rightarrow 3^{100} \times 2^{50} \equiv 1 \times -1 \pmod{5} \equiv -1 \pmod{5} \]

but \[ -1 \equiv 4 \pmod{5} \]

\[ \therefore 3^{100} \times 2^{50} \equiv 4 \pmod{5} \]
28) If $a$ and $b$ are positive integers such that $a^2 - b^2$ is a prime number, then $a^2 - b^2$ is

1) $a+b$
2) $a - b$
3) $ab$
4) 1
Ans: is 1

\[ a^2 - b^2 = (a+b)(a-b) \] is a prime.

\[ \therefore (a+b)(a-b) \] is divisible by 1 or its self. But \( a - b < a+b \) \( \therefore a-b=1 \)

\[ \therefore a^2 - b^2 = a+b \]
29) Which of the following is a prime number?

1) 370261
2) 1003
3) 73271
4) 667
Ans: is 1

$\frac{17}{1003}, \frac{11}{73271} \& \frac{29}{667}$. but none of the prime & less than 608 divides the first No.
30) Which of the following is false?

1) An odd number is relatively prime to the next even number.
2) $3x \equiv 4 \pmod{6}$ has solution.
3) $ax \equiv bx \pmod{m} ; x \neq 0 \Rightarrow a \equiv b \pmod{m}$
4) $a.x + b.y = d \Rightarrow (a, b) = d$
Ans: is 2

(3,6) = 3 but 3 does not divides 4

∴ no solution.

Remaining are all known results

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31) For all positive values of p, q, r, and s, 

\[
\frac{(p^2 + p + 1)(q^2 + q + 1)(r^2 + r + 1)(s^2 + s + 1)}{pqrs}
\]

will not be less than 

1) 81 
2) 91 
3) 101 
4) 111
Ans: is 1

\[ \frac{p^2 + p + 1}{p} = p + 1 + \frac{1}{p} \geq 3 \quad (\because \text{p is positive integer}) \]

Similarly,

\[ \frac{q^2 + q + 1}{q} = q + 1 + \frac{1}{q} \geq 3 \quad \text{etc.} \]

\[ \therefore \text{given expression is } \geq 3 \times 3 \times 3 \times 3 = 81. \]

\[ \therefore \text{expression cannot be less than 81.} \]
32) If \((a+b)^n \equiv x \pmod{a}\), then \((n\) is a +ve integer)

1) \(x = a^2\)  
2) \(x = a^n\)  
3) \(x = b^n\)  
4) none of these
Ans: is 3

\[(a+b)^n = a^n + \sum_{k=1}^{n} \binom{n}{k} a^{n-k} b^k + b^n\]

\[\therefore (a+b)^n - b^n = a \left[ a^{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} a^{n-k-1} b^k \right]
\]

\[(a+b)^n - b^n = ak \text{ where } k \in \mathbb{Z}.
\]

\[\therefore a/[(a+b)^n - b^n]
\]

\[\Rightarrow (a+b)^n \equiv b^n \pmod{a}
\]

\[\therefore x = b^n
\]
33) If \( 27 = 189m + 24n \) then \( m \) & \( n \) are

1) unique
2) not unique
3) prime numbers
4) none of these
Ans : is 2

If \((a,b) = d \Rightarrow d = ax + by\)

where \(x, y \in \mathbb{Z}\). Here \(x, y\) are not unique.
34) If $2x \equiv 3 \pmod{7}$, then the values of $x$ such that $9 \leq x \leq 30$ are

1) 12, 19, 26
2) 11, 18, 25
3) 10, 17, 24
4) None of these
Ans: is 1

The soln. is \( x \equiv 5 \pmod{7} \)

\[ \therefore \text{Soln. set is } \{ \ldots 2, 5, 12, 19, 26, 33, \ldots \} \]

\[ \therefore \text{required values of } x \text{ are } 12, 19, 26. \]
35) If $p$ is a prime number and $P$ is the product of all prime numbers less than or equal to $p$, then

1) $P - 1$ is a prime
2) $P + 1$ is not a prime number
3) $P + 1$ is a prime number
4) $P + 1$ is a composite number
Ans: is 3

Known result while proving the thm. The primes are infinite.
36) \(4x + 9 \equiv 3 \pmod{5}\) can be written as

1) \(x \equiv 5 \pmod{6}\)
2) \(x \equiv 3 \pmod{15}\)
3) \(x \equiv 6 \pmod{15}\)
4) None of these
Ans : is 3

when \( x=6 \), \( 4 \cdot 6 + 9 = 33 \equiv 3 \pmod{5} \)

it satisfies the given congruence.

Hence (3) is right answer
37) If \((3-x) \equiv (2x-5) \pmod{4}\), then one of the values of \(x\) is

1) 3
2) 4
3) 18
4) 5

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Ans : is 2

3-x-2x+5 = -3x+8 is divisible by 4

when x=4, -3 (4)+8 = -4 is divisible by 4.
38) The remainder when 64x65x66 is divided by 67 is

1) 60
2) 61
3) 62
4) 63
Ans: is 2

\[ 64 \times 65 \times 66 \equiv (-3) (-2) (-1) \pmod{67} \]
\[ \equiv -6 \pmod{67} \]
\[ \equiv 61 \pmod{67} \]
GROUPS

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GROUP

1) If x, y, z are three elements of a group and then $(xy^{-1}z)^{-1} =$

1) $x^{-1}y^{-1}z^{-1}$
2) $x^{-1}yz$
3) $z^{-1}yx^{-1}$
4) $(xy^{-1}z)^{-1}$
Ans: is 3 since $(a \star b)^{-1} = b^{-1} \star a^{-1}$.

Question is just extension of this property.
2) If \( a \ast b = \sqrt{a} + \sqrt{b} \), then \( \ast \) is a binary operation on

1) \( \mathbb{R} \)
2) \( \mathbb{Q}^+ \)
3) \( \mathbb{R}_0 \)
4) \( \mathbb{R}^+ \)
Ans: is 4
if $a = -1, b = 3$ then

$$a \star b = \sqrt{1 + \sqrt{3}} \in \mathbb{C}$$
3) The identity element of $a \star b = a^{b-1}$ is

1) 1
2) 0
3) 2
4) $-1$

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Ans: \( \text{is 3} \)

\[
a \times e = a \Rightarrow a^{e-1} = a
\]

\[
\Rightarrow e - 1 = 1 \Rightarrow e = 2
\]
4) In the group of rational numbers under a binary operation \( \ast \) defined by \( a \ast b = a+b-1 \) then identity element is

1) 1
2) 0
3) 2
4) -1
Ans : is 1

\[ a \times e = a \Rightarrow a + e - 1 = a \]

\[ \therefore e - 1 = 0 \Rightarrow e = 1 \]
5) The set $G = \{ -3, -2, -1, 0, 1, 2, 3 \}$ w.r.t. addition does not form a group since.

1) The closure axiom is not satisfied
2) The associative axiom is not satisfied
3) The commutative axiom is not satisfied
4) Identity axiom is not satisfied
Ans: is 1 since 2, 3 ∈ G but 2 + 3 = 5 ∉ G
6) If \( a \star b = 2a - 3b \) on the set of integers. Then \( \star \) is

1) Associative but not commutative
2) Associative and commutative
3) A binary operation
4) Commutative but not associative
Ans: is 3

∀ a, b ∈ Z, a*b = 2a - 3b ∈ Z

(i.e., if a = 1, b = -2 then

2 * 1 - 3 * (-2) = 2 + 6 = 8 ∈ Z )
7) In the multiplicative of cube roots of unity the inverse of $w^{99}$ is

1) $w$
2) 1
3) $w^2$
4) Does not exist.

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Ans : is 2

\[ \text{Ans : is 2} \]

\[ W^3 = 1 \]

\[ \therefore (W^3)^{33} = 1 \]
8) The incorrect statement is

1) In $(G, .) \ ab = ac \Rightarrow b = c, \ \forall \ a, b, c \in G$
2) Cube roots of unity form an abelian group under addition
3) In a abelian group $(ab)^3 = a^3b^3, \ \forall a, b \in G$
4) In a group of even order, there exists at least two elements with their own inverse.
Cube roots of unity; $1, w, w^2$

form an abelian group under multiplication.

Ans : is 2
9) If H & K are two subgroups of a group G, then identify the correct statement:

1) $H \cap K$ is a sub group
2) $H \cup K$ is a sub group
3) Neither $H \cup K$ nor $H \cap K$ is sub group
4) Nothing can be said about $H \cup K$ and $H \cap K$
Ans: is 1

Let $H = \{0, 2, 4\}$, $K = \{0, 3\}$ are subgroups of $G = \{0, 1, 2, 3, 4, 5\}$ under $+_6$

i.e., $H \cup K = \{0, 2, 3, 4\}$ is not closed

i.e., $2+3=5 \notin H \cup K$
10) In the group $G = \{e, a, b\}$ of order 3, $a^5b^4$ is

1) 3
2) ab
3) a
4) b
Ans: is 3

\[ ab = e \Rightarrow (ab)^4 = e \]

i.e. \( a (a^4b^4) = ae \)

\[ \Rightarrow a^5b^4 = a \]
11) In a group \((G, \ast)\), \(a \ast x = b\) where \(a, b \in G\) has

1) Unique solution
2) No solution
3) More than one solution
4) Infinite number of solutions
Ans: is 1

\[ a \star x = b \Rightarrow a^{-1} \star (a \star x) = a^{-1} \star b \]

\[ (a^{-1} \star a) \star x = a^{-1} \star b \Rightarrow x = a^{-1} \star b \]
12) The set of (non singular) matrices of order $2 \times 2$ over $\mathbb{Z}$ under matrix multiplication is

1) Group
2) Semi group
3) Abelian group
4) Non-abelian group
Ans: is 2

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M \]

When \( A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \), \( |A| = 2 \) but \( A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} \)

but \( \frac{1}{2} \notin \mathbb{Z} \)
13) Which of the following is a subgroup of \( G = \{0, 1, 2, 3, 4, 5\} \) under addition modulo 6

1) \( \{0, 2\} \)
2) \( \{0, 1\} \)
3) \( \{0, 4\} \)
4) \( \{0, 3\} \)
Ans : is 4
2 + 6 \cdot 2 = 4 \in \{0, 2\} \text{ etc.,}
but 3 + 6 \cdot 3 = 0
14) The set of integers is

1) Finite group
2) Additive group
3) Multiplicative group
4) None of these
Ans : is 2
15) The set of all integers is not a group under multiplication because

1) Closure property fails
2) Associative law does not hold good
3) There is no identity element
4) There is no inverse
Ans : is 4
Inverse 0 does not exists
(also $2 \in \mathbb{Z}$ but $2^{-1} = \frac{1}{2} \notin \mathbb{Z}$)
16) A subset $H$ of a group $(G, \cdot)$ is a subgroup of $G$ iff:

1) $a, b \in H \Rightarrow a \cdot b \in H$
2) $a \in H \Rightarrow a^{-1} \in H$
3) $a, b \in H \Rightarrow a \cdot b^{-1} \in H$
4) $H$ contains identity of $G$. 
Ans: is 3
By thm.
17) \( Z_n = \{0, 1, 2, \ldots, (n-1)\} \) fails to be a group under multiplication modulo \( n \) because

1) Closure property fails
2) Closure holds but not associativity
3) There is no identity
4) There is no inverse for an element of the set
Ans: is 4

at least for one element ‘0’

has no inverse in \( \mathbb{Z}_n \).
18) \( G = \{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \neq 0 \& x \in \mathbb{R} \} \) is an abelian group under matrix multiplication. Then the identity element is

1) \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)
2) \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \)
3) \( \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \)
4) \( \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \)
\[ A(x) = \begin{bmatrix} x & x \\ x & x \end{bmatrix}, A(e) = \begin{bmatrix} e & e \\ e & e \end{bmatrix} \]

if

\[ A(x) \cdot A(e) = A(x) \text{ then} \]

\[ \begin{bmatrix} 2xe & 2xe \\ 2xe & 2xe \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \Rightarrow 2xe = x \]

\[ \Rightarrow e = \frac{1}{2} : \ A(e) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \]
19) In the group $G = \{3, 6, 9, 12\}$ under $\times_{15}$, the identity is

1) 3
2) 6
3) 9
4) 12
Ans: is 2

Since $3 \times 156 = 3$, $6 \times 156 = 6$, $9 \times 156 = 9$ etc.,
20) The set of all $2 \times 2$ matrices over the real numbers is not a group under matrix multiplication because

1) Inverse law is not satisfied
2) Associative law is not satisfied
3) Identity element does not exist
4) Closure law is not satisfied
Ans : is 1

If A is a singular matrix
of 2 x 2 order matrix then
A\(^{-1}\) does not exist.
21) \((\mathbb{Z}, \star)\) is a group with \(a \star b = a+b+1\), \(\forall a, b \in \mathbb{Z}\). The inverse of \(a\) is

1) \(A+2\)
2) \(-a+2\)
3) \(-a-2\)
4) \(a-2\)
Ans: is 3

\[ a \ast e = a \Rightarrow a + e + 1 = a \Rightarrow e = -1 \]

\[ a \ast a^{-1} = e \Rightarrow a + a^{-1} + 1 = -1 \]

\[ \Rightarrow a^{-1} = -2 - a \]
22) The four matrices \(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\) under multiplication form is

1) a group
2) a semi group
3) an abelian group
4) infinite group
Ans : is 3

Taking them as I, A, B, C

then AB=C, BC=A, etc., & A.I=A etc.

Also, A.A=I ⇒ A⁻¹=A || ly B⁻¹=B,

C⁻¹=C also AB=BA
23) In the group \((G, \ast)\), \(a \ast b = \frac{ab}{5}\) where \(\forall a, b \in G\). The identity and inverse of 8 are respectively.

1) 5 & \frac{5}{3}
2) 5 & \frac{25}{8}
3) 5 & \frac{8}{25}
4) 5 & \frac{3}{5}
Ans: is 2

\[ a \ast e = a \Rightarrow ae/5 = a \Rightarrow e = 5 \]

& \[ a \ast a^{-1} = e \Rightarrow aa^{-1} = 5 \Rightarrow a^{-1} = \frac{25}{a} \]

\[ \therefore 8^{-1} = \frac{25}{8} \]
24) The proper subgroups of the group \( G = \{0, 1, 2, 3, 4, 5\} \) under addition modulo 6 are:

1) \( \{0, 3\} \) and \( \{0, 2, 4\} \)
2) \( \{0, 1, 3\} \) and \( \{0, 1, 4\} \)
3) \( \{0, 1\} \) and \( \{3, 4, 5\} \)
4) \( \{0\} \) and \( \{0, 1, 2, 3, 4, 5\} \)
Ans : is 1
Since \(0(G)=6\) & \(6=2 \times 3\)
\[\therefore \text{It has proper subgroups of orders } 2 \text{ & } 3\]
In (1) \(3+63=0\) & \(2+62=4, 4+62=0\)
\(4+64=2\) all in the sets
25) In the group $G = \{1, 3, 7, 9\}$ under multiplication modulo 10, the value of $(3 \times_{10} 7^{-1})^{-1}$ is

1) 5
2) 3
3) 7
4) 9
Ans: is 4

e = 1

7 \times_{10} 3 = 1 \Rightarrow 7^{-1} = 3

\therefore 3 \times_{10} 3 = 9

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26) The incorrect statement is

1) The identity element in a group is unique
2) In a group of even order, there exists an element \( a \neq e \) such that \( a^2 = e \).
3) The cube roots of unity are \( 1, \frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2} \).
4) In an abelian group \( (ab)^2 = a^2b^2, \forall a, b \in G \).
Cube roots of unity are

\[1, w = \frac{-1 + i\sqrt{3}}{2}, \quad w^2 = \frac{-1 - i\sqrt{3}}{2}\]

Ans: is 3
27) In the multiplicative group of fourth roots of unity the inverse of $i^{103}$ is

1) 1
2) $-1$
3) $i$
4) $-i$
Ans: is 3

e = 1

\[ i^{103} = i^{100} \cdot i^3 = (i^4)^{25} \cdot (i^2)i = 1 \cdot (-1) \cdot i = -i \]

\therefore \text{inverse of} \ -i \ \text{is} \ i.
28) Let \( \mathbb{Q}_1 = \mathbb{Q} - \{1\} \) be the set of all rationals except 1 and \( \star \) is defined as \( a \star b = a + b - ab \quad \forall \ a, b \in \mathbb{Q}_1 \). The inverse of 2 is

1) 2
2) 1
3) 0
4) – 2
Ans : is 1
\[ a \times e = a \Rightarrow a + e - ae = a \]
\[ \Rightarrow e(1-a) = 0 \Rightarrow e = 0 \quad (\because a \neq 1 \notin Q_1) \]
\[ & a \times a^{-1} = e \Rightarrow a + a^{-1} - aa^{-1} = 0 \]
\[ \Rightarrow a^{-1}(1-a) = -a \Rightarrow a^{-1} = \frac{-a}{1-a} \]
\[ (\because 1-a \neq 0) \]
\[ \therefore 2^{-1} = \frac{-2}{1-2} \Rightarrow 2^{-1} = 2 \]
29) In the group \( \{\mathbb{Z}_6, + \ (\text{mod } 6)\} \),
\[2 + 4^{-1} + 3^{-1}\] is equal to

1) 2
2) 1
3) 4
4) 3

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Ans: is 2

\[ e = 0 \]

\[ 2 + \sigma 4^{-1} + \sigma 3^{-1} = 2 + \sigma 2 + \sigma 3 = 1 \]
30) Every group of order 7 is

1) Not abelian
2) Not cyclic
3) Cyclic
4) None of these
Ans: is 3

Every group of prime order is cyclic

7 is prime
31) If \( g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \) and \( h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \) are two permutations in group \( S_4 \), then \((h \times g)(2) = \) 

1) 2  
2) 1  
3) 3  
4) 4
Ans: is 2

\[(hxg)^2 = h[g(2)] = h(3) = 1\]
32) If \( g = (1 \ 2 \ 3 \ 4) \) then \( g^{-1} \)

1) \( (3 \ 4 \ 1 \ 2) \)

2) \( (4 \ 2 \ 1 \ 3) \)

3) \( (1 \ 2 \ 3 \ 4) \)

4) \( (3 \ 1 \ 4 \ 2) \)
Ans: is 1

\[ g^{-1} = (3 \ 4 \ 1 \ 2) \]

\[ g^{-1} = (1 \ 2 \ 3 \ 4) \]
33) In the group \{1, 2, 3, 4, 5, 6\} under multiplication modulo 7, 5x=4 has the solution x = 

1) 0.8
2) 2
3) 3
4) 5
Ans : is 4

(e=1)

5x \cdot 3 = 1 \Rightarrow 5^{-1} = 3

\therefore 5x = 4 \Rightarrow x = 5^{-1}x \cdot 4 = 3 \cdot 4 = 5
34) In the group $G=\{2, 4, 6, 8\}$ under $X_{10}$, the inverse of 4 is

1) 6
2) 8
3) 4
4) 2

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Ans: is 3

Here $e=6$ since $4x_{10}6=4$ etc.

$\therefore 4x_{10}4=6 \iff 4^{-1}=4$
35) The Set \{-1, 0, 1\} is not a group w.r.t. addition because it does not satisfy

1) Closure property
2) Associative law
3) Existence of identity
4) Existence of inverse

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Ans: is 1

$1+1=2 \in \text{the set}$
36) If every element of a group G is its own inverse, then G is

1) Finite
2) Infinite
3) Cyclic
4) Abelian
Ans: is 4

since \( a = a^{-1}, \ b = b^{-1} \ \forall \ a, b \in G \)

Now \( (ab)^{-1} = ab \) (by hypothesis)

\[ \Rightarrow b^{-1}a^{-1} = ab, \ by \ property \]

\[ \Rightarrow ba = ab \]

\[ \therefore \ G \ is \ abelian \]
37) If a, b, c, are three elements of a group \((G, \star)\), and \((a \star b) \star x = c\), then \(x =\)

1) \(c \star (a^{-1} \star b^{-1})\)
2) \(c \star (b^{-1} \star a^{-1})\)
3) \((b^{-1} \star c^{-1}) \star c\)
4) \((a^{-1} \star b^{-1}) \star c\)
Ans : is 3

\[(a \ast b)^{-1} \ast (a \ast b) \ast x = (a \ast b)^{-1} \ast c\]

\[e \ast x = (b^{-1} \ast a^{-1}) \ast c\]
38) If \( \{ z_7, x_7 \} \) is a group, then the inverse of 6 is

1) 6
2) 4
3) 1
4) 3
Ans: is 1

since $6 \cdot 6 \equiv 36 \equiv 1 \pmod{7}$

where 

$s_6 = 1$

∴ $6^{-1} = 6$